Contrat Doctoral — ED Galilée

Titre du sujet : Exposants de Lyapunov des fractions continues multidimensionnelles

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Continued fraction type expansions provide increasingly good rational approximations of real numbers. A multidimensional continued fraction is expected to yield simultaneous better and better rational approximations with the same denominator for given d-tuples of real numbers. More precisely, it has to produce a sequence of positive integers $(q_n)_n$ such that the distance to the nearest integer $\|q_n(\alpha_1, \cdots, \alpha_d)\|_{\mathbb{Z}^d}$ converges exponentially fast to 0 with respect to q_n, and ideally in $q_n^{-\frac{1}{d}}$, with respect to Dirichlet's theorem.

However, there is no canonical extension of regular continued fractions to higher dimensions, and the zoology of existing types of algorithms is particularly rich. Note that regular continued fractions rely on Euclid's algorithm : starting with two numbers, one subtracts the smallest from the largest. If we start with at least three numbers, it is not clear how to decide which operation has to be performed on these numbers in order to get something analogous to Euclid's algorithm, hence the diversity and multitude of existing generalizations.

We can roughly classify the main generalizations as follows : the ones based on convex hulls defined in terms of Klein polyhedra and Arnold sails [Kar09], the ones issued from best approximations [Che13], and the ones expressed as piecewise linear fractional transformations, i.e., as unimodular matrices acting on lattice bases [Sch00]. Klein polyhedra are well suited for periodic trajectories and algebraic number fields by providing generalized Lagrange's theorem. For best approximations approaches, the sequence heavily depends on the chosen norm and the associated transformations are no longer unimodular. Nevertheless, one can use the action of the diagonal flow on the space of unimodular lattices [KLDM07, CC].

We focus on unimodular continued fractions [Lag93, Sch00]. Unimodular means that one works with matrices in $GL(d, \mathbb{Z})$, (i.e., matrices with integer entries with determinant ± 1). More precisely, a d-dimensional unimodular continued fraction algorithm associates with $\alpha = (\alpha_1, \cdots, \alpha_{d-1}) \in [0,1]^{d-1}$ a sequence of matrices $(A^{(n)})_{n\in\mathbb{N}}$ with values in $GL(d, \mathbb{Z})$, via two maps $T : [0,1]^{d-1} \to [0,1]^{d-1}$ (the dynamical system), $A : [0,1]^{d-1} \to GL(d, \mathbb{Z})$ (the cocyle map), wit $A^{(n)} = A(T^n)(\alpha)$ for all n. Matrices $A^{(n)}$ play the role of partial quotients and the matrices $A^{(1)} \cdots A^{(n)}$ produce convergents. These latter products provide Diophantine approximations (via their column vectors) of the direction $(\alpha, 1)$ by points of the lattice $\mathbb{Z}^d$. Such algorithms are said Markovian, or without memory. Indeed, the $(n+1)$th step of the algorithm only depends on the map T and on the value $T^n(\alpha)$, contrary for example to lattice reduction or LLL algorithms, such as developed e.g. in [FF79, Fer87, Lag85, Jus92, BS13].

Famous examples of the unimodular approach are the Jacobi-Perron, Brun or Selmer algorithms. The main advantage of these classical unimodular continued fractions is that they can be expressed as dynamical systems whose ergodic study has already been well understood [Sch00]. The main disadvantage relies in the quality of approximation in higher dimension. It is governed by Lyapounov exponents which describe the asymptotic behaviour of the singular values of large products of random matrices, under the ergodic hypothesis. The approximation exponent can be expressed as $1-\frac{\lambda_2}{\lambda_1}$ according to [Lag93] (λ_1 and λ_2 being the two largest Lyapunov exponents of the associated dynamical system), implying the exponential convergence to zero of the distance to integers mentioned in the beginning, also called strong convergence, when $\lambda_2 < 0$. Optimal convergence, with exponential speed given by Dirichlet's exponent $1/d$, is equivalent to having a completely degenerate Lyapunov spectrum, with $\lambda_2 = \ldots = \lambda_{d+1} = -\lambda_1/d$.

It has been recently noticed experimentally in [BST21] that λ_2 is not even negative in higher dimension for the most classical unimodular algorithms such as the Jacobi-Perron, Brun or Selmer algorithms. **The aim of this subject is first to confirm numerically, heuristically and theoretically these estimates, and secondly to design strong convergent continued fraction algorithms in any dimension.**

The approach we plan to adopt here for the study and the design of continued fractions relies mostly on the formalism developed by C. Fougeron in [Fou20], inspired by Rauzy induction. The generic behaviour of such a continued fraction algorithm is described here as a random walk with memory recorded by a finite dimensional vector acting on a finite graph. Ergodic properties are then described in terms of combinatorial properties of these graphs. In particular, it provides a unified proof of ergodicity for classical examples (Brun, Selmer and Arnoux-Rauzy-Poincare algorithm), as well as new results such as uniqueness of the measure of maximal entropy on a canonical suspension. These concepts also bring a new perspective to some fractal sets (such as Rauzy gaskets [AHS16]) by providing general explicit upper bound on Hausdorff dimensions of fractals described in this formalism, as well as a construction of their measure of maximal entropy.

The methodology employed for the study of Lyapunov exponents will proceed as follows.

Initially, emphasis will be placed on lower dimensions. Numerical experiments will be conducted using SageMath, leveraging the formalism mentioned above conducive to such investigations. This formalism translates these algorithms into a manageable framework, treating them as random walks on labeled graphs, thus facilitating computational tractability.

Furthermore, seminal research by Palley and Ursell in the 1930s [PU30], establishes the positivity of the second Lyapunov exponent for the Selmer algorithm in dimension 2. This work has been extended more recently by Schratzberger [Sch98] and Broise-Guivarc'h [BAG01] for other algorithms.

The investigation will also extend to specific matrix families, such as those generated by the Jacobi-Perron algorithm. The impact of the choice of the measure governing these matrix products will be considered. Special attention will be given to the influence of the dimension on the Lyapunov exponents of these algorithms. Additionally, fractals described within this formalism will be analyzed in terms of their Hausdorff dimensions.

Beyond analysis, this formalism will inform the design of strongly convergent algorithms. One approach involves deriving continued fraction algorithms from lattice reduction algorithms, leveraging their capacity to compute short vectors and achieve Dirichlet's bound, up to a constant depending exponentially on the dimension. Such algorithms are made of a succession of permutations and subtractions. The decisions are taken for classical unimodular continued fractions by comparing entries, whereas lattice reduction involves quadratic comparisons. Several attempts already exist in this direction. Let us quote [Lag85], [Lag94] based on Minkowski reduced forms and [Che13, Beu14, BS13] built on LLL. However, they do not present the same advantages as more classical memoryless algorithms, in particular concerning effectiveness in the production of sequences of convergents.

Encadrement

La thèse sera menée principalement au LAGA (Laboratoire d'Analyse, Géométrie et Applications, UMR 7539) sous les directions conjointes de J. Barral (PR, Institut Galillée, Université Sorbonne Paris Nord), V. Berthé (DR CNRS, IRIF, Université Paris Cité) et C. Fougeron (MdC, Institut Galillée, Université Sorbonne Paris Nord).

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